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POST-BIFURCATION AND IMPERFECTION SENSITIVITY ANALYSIS OF PLASTIC CLAMPED CIRCULAR PLATES

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Abstract—The plastic post-bifurcation and imperfection sensitivity behaviors of clamped circular plates are analyzed using the systematic perturbation method. It is found that there is a second-order term of post-bifurcation deflection expanded in terms of the eigenmodal amplitude, but no corresponding term for elastic postbuckling. High order post-bifurcation terms are calculated to improve the analysis. The influence of geometrical imperfections on the load-bearing capacity of clamped circular plates is expressed analytically. The results of using this method are compared with those obtained by the finite element method.

NOTATION

a_n, b_n	parameters appearing in eqns (36)–(38)
d	end point of the unloading interval along the thickness
$E, \bar{E}, E_t$	Young's modulus, effective modulus, tangent modulus
$e_{\alpha\beta}$	strains
J_n	nth-order Bessel function
k	first zero point of J_1
$L_{ijkl}, \bar{L}_{\alpha\beta\mu\nu}$	plastic moduli, their plane stress expressions
$\mathscr{L}_{ijkl}, \bar{\mathscr{L}}_{\alpha\beta\mu\nu}$	elastic moduli, their plane stress expressions
$M_{\alpha\beta}$	bending moment
m	a parameter defined in eqn (49)
$N_{\alpha\beta}$	resultant stresses
$n_{ij}, \bar{n}_{\alpha\beta}$	normal of the loading surface, its plane stress expression
R	plate radius
r	radial coordinate
t	plate thickness
u	radial displacement
w	plate deflection
$\bar{w}$	plate imperfection
Y_n	nth-order Neumann function
z	lateral coordinate
β	a parameter defined in eqn (39)
δ	length of unloading interval along the thickness
ε	imperfection amplitude
ζ	a parameter quantifying the unloading area
η	eigenmodal amplitude
$\nu, \bar{\nu}$	Poisson's ratio, effective Poisson's ratio
ξ	coordinate variables of inner field
$\bar{\xi}$	coordinate variable of outer field
$\hat{\xi}$	deflection amplitude at first unloading
σ	loading parameter
$\sigma_{\alpha\beta}$	stresses
$\sigma_c, \hat{\sigma}$	bifurcation load, first unloading load
σ_e, σ_y	equivalent stress, yield stress
$\bar{L}^c_{\alpha\beta\mu\nu}, \bar{m}_c, \bar{\nu}_c, \bar{E}_c, \ldots$	quantities calculated at bifurcation
$\hat{\bar{L}}_{\alpha\beta\mu\nu}, \hat{\bar{m}}, \hat{\bar{\nu}}, \hat{\bar{E}}, \ldots$	quantities calculated at first unloading
$\overset{(I)}{w}, \overset{(I)}{u}, \overset{(I)}{\dot{N}}_{\alpha\beta}, \overset{(I)}{\dot{M}}_{\alpha\beta}, \ldots$	quantities of inner field
$\overset{(II)}{w}, \overset{(II)}{u}, \overset{(II)}{\dot{N}}_{\alpha\beta}, \overset{(II)}{\dot{M}}_{\alpha\beta}, \ldots$	quantities of outer field
$w_n, u_n, d_n, \ldots$	nth-order terms of $w, u, \delta, \ldots$

1. INTRODUCTION

It is a common observation that clamped circular plates will bifurcate after they are deformed in the plastic range. In earlier studies, Hutchinson (1974) made post-bifurcation analyses on the basis of his post-bifurcation theory. In his plastic post-bifurcation theory, he provided a clear picture of the post-bifurcation behaviors of structures in the plastic range for the first time. His work, however, did not include the imperfection sensitivity and higher order asymptotic analysis. Needleman (1975) calculated the post-bifurcation behavior both analytically and numerically, but the imperfection sensitivity was examined only numerically.

This paper presents a new systematic perturbation method of analyzing the post-bifurcation and imperfection sensitivity of clamped circular plates. The main feature of the analysis is the use of a matched asymptotic expansion of fields inside and outside the unloading zone, the boundary of which changes with the increase of applied loading. The radius of the unloading circle on one of the outer surfaces of the plate was used as an expansion parameter. Higher order asymptotic post-bifurcation and imperfection sensitivity analyses were carried out. New aspects of the plastic post-bifurcation were also explored.

The systematic perturbation method is an improvement on the asymptotic method used by Hutchinson in his general plastic post-bifurcation theory. A general presentation of the method can be found in Su (1988) and Su and Lu (1990a,b).

2. BASIC EQUATIONS

As shown in Fig. 1, a clamped circular plate with radius R and thickness t is subjected to a homogeneous loading, $-\sigma t$. A cylindrical coordinate system is established, with the polar plane (r, θ) lying on the midplane and the z-axis normal to it. Here, attention is restricted to axisymmetrical deformations. The radial displacement is denoted by $u(r)$, and the deflection by $w(r)$. The strain rates $\dot{e}_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$ are

$$\begin{cases} \dot{e}_{11} = \dot{e}_r = \dot{u}_{,r} + w_{,r}\dot{w}_{,r} - z\dot{w}_{,rr} \\ \dot{e}_{22} = \dot{e}_\theta = \dot{u}/r - z\dot{w}_{,r}/r \\ \dot{e}_{12} = \dot{e}_{21} = \dot{e}_{r\theta} = 0. \end{cases} \tag{1}$$

If J_2 flow theory is then adopted, the constitutive relation is

$$\dot{\sigma}_{\alpha\beta} = \begin{cases} \bar{\mathscr{L}}_{\alpha\beta\mu\nu}\dot{e}_{\mu\nu} & \text{for unloading} \quad (2) \\ \bar{L}_{\alpha\beta\mu\nu}\dot{e}_{\mu\nu} & \text{for loading,} \quad (3) \end{cases}$$

where $\bar{\mathscr{L}}_{\alpha\beta\mu\nu}$ and $\bar{L}_{\alpha\beta\mu\nu}$ are the elastic moduli and plastic moduli in the plane stress expression, respectively. They are related to their three-dimensional counterparts by

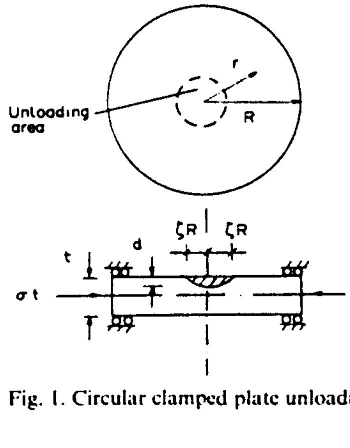

Fig. 1. Circular clamped plate unloading area.

$$\bar{\mathscr{L}}_{\alpha\beta\mu\nu} = \mathscr{L}_{\alpha\beta\mu\nu} - \mathscr{L}_{\alpha\beta33}\mathscr{L}_{\mu\nu33}/\mathscr{L}_{3333}, \quad \bar{L}_{\alpha\beta\mu\nu} = L_{\alpha\beta\mu\nu} - L_{\alpha\beta33}L_{\mu\nu33}/L_{3333}.$$

For the isotropic material now considered, we have

$$\mathscr{L}_{ijkl} = \frac{E}{1+\nu}\left[\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right] \quad (i,j,k,l = 1,3) \tag{4}$$

$$L_{ijkl} = \mathscr{L}_{ijkl} - \frac{3}{2}\frac{E}{1+\nu}\frac{1/E_t - 1/E}{[\frac{2}{3}(1+\nu)/E + 1/E_t - 1/E]\sigma_e^2}, \tag{5}$$

where E_t is the tangent slope of the uniaxial stress–strain curve, and

$$\sigma_e^2 = \tfrac{3}{2}s_{ij}s_{ij}, \quad s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}.$$

The rates of the resultant stress $\dot{N}_{\alpha\beta}$ and bending moment $\dot{M}_{\alpha\beta}$ are defined as

$$\dot{N}_{\alpha\beta} = \int_{-t/2}^{t/2}\dot{\sigma}_{\alpha\beta}\,\mathrm{d}z, \quad \dot{M}_{\alpha\beta} = \int_{-t/2}^{t/2}\dot{\sigma}_{\alpha\beta}z\,\mathrm{d}z. \tag{6}$$

The rate form of the virtual work principle is

$$\int_0^R [\dot{N}_r\delta u_{,r} + \dot{N}_\theta\delta u/r - \dot{M}_r\delta w_{,rr} + (\dot{N}_r w_{,r} - \dot{M}_\theta/r + N_r\dot{w}_{,r})\delta w_{,r}]r\,\mathrm{d}r = \dot{\sigma}t\delta u(R). \tag{7}$$

A perfect plate deforms homogeneously with $\sigma_r = \sigma_\theta = \sigma$ before bifurcation. Its plastic constitutive relation can be expressed as

$$\dot{\sigma}_r = \frac{\bar{E}}{1-\bar{\nu}^2}(\dot{e}_r + \bar{\nu}\dot{e}_\theta), \quad \dot{\sigma}_\theta = \frac{\bar{E}}{1-\bar{\nu}^2}(\dot{e}_\theta + \bar{\nu}\dot{e}_r) \tag{8}$$

where $\bar{E}$ and $\bar{\nu}$ are the effective Young's modulus and Poisson's ratio, respectively:

$$E/\bar{E} = 1 + (E/E_t - 1)/4, \quad \bar{\nu} = (\bar{E}/E)[\nu - (E/E_t - 1)/4]. \tag{9}$$

The plastic bifurcation load and eigenmode of the plate are

$$\sigma_c = k^2\bar{E}_c t^2/[12(1-\bar{\nu}_c^2)R^2] \tag{10}$$

$$w_c = -[J_0(kr) - J_0(k)]/[1 - J_0(k)], \tag{11}$$

where J_n is the nth-order Bessel function, k is the first zero point of J_1, and $\bar{E}_c$, $\bar{\nu}_c$ are $\bar{E}$, $\bar{\nu}$ calculated with $\sigma_r = \sigma_\theta = \sigma_c$, respectively. The minus sign before the eigenmode in eqn (11) is used to make sure that the plate bifurcates downwards. According to the plastic bifurcation theory (Hutchinson, 1974), the plate will unload at the point $(0, 0, -t/2)$ when it also bifurcates. In the process of post-bifurcation loading, the unloading will extend from the point $(0, 0, -t/2)$ in both the r and z directions. As the unloading is axisymmetric, the intersection of the unloading area and a plane parallel to the midplane of the plate will be a circular region around the center, if the two are intersected. Of all these unloading regions, the one with radius ζR on the lower outer surface $z = -t/2$ will be the largest. For the part of the plate with $r \geqslant \zeta R$, the plate deforms plastically through the thickness. Equation (6) therefore becomes

$$\overset{\text{(II)}}{\dot{N}}_{\alpha\beta} = \int_{-t/2}^{t/2} \bar{L}_{\alpha\beta\mu\nu}\dot{e}_{\mu\nu}\,\mathrm{d}z, \quad \overset{\text{(II)}}{\dot{M}}_{\alpha\beta} = \int_{-t/2}^{t/2} \bar{L}_{\alpha\beta\mu\nu}\dot{e}_{\mu\nu}z\,\mathrm{d}z \tag{12}$$

where the symbol "II" represents the part of $r \geqslant \zeta R$ which is referred to as the outer field in the following statement. In the portion of the plate with $r \leqslant \zeta R$, referred to as the inner field, there is an unloading interval along the z-axis for every specific r. The unloading interval will be $z \in [-t/2, d]$, where d is defined by the neutral loading condition

$$[\bar{n}_{\alpha\beta}(r,z)\dot{e}_{\alpha\beta}(r,z)]|_{z=d} \doteq \bar{n}^{c}_{\alpha\beta}\dot{e}_{\alpha\beta}(r,d) = 0, \tag{13}$$

where $\bar{n}_{\alpha\beta}$ is the plane stress expression of the normal of the loading surface, $\bar{n}^{c}_{\alpha\beta}$ is the $\bar{n}_{\alpha\beta}$ when the plate bifurcates and $\bar{n}^{c}_{\alpha\beta} = -\delta_{\alpha\beta}$ in the present case. Equation (13) then leads to

$$\dot{u}_{,r} + w_{,r}\dot{w}_{,r} + \dot{u}/r - d(r)[\dot{w}_{,rr} + \dot{w}_{,r}/r] = 0. \tag{14}$$

Since those points of $z \in [-t/2, d]$ deform elastically, and those of $z \in [d, t/2]$ plastically, we have

$$\left.\begin{aligned}\overset{\text{(I)}}{\dot{N}}_{\alpha\beta} &= \int_{-t/2}^{d} \bar{\mathscr{L}}_{\alpha\beta\mu\nu}\dot{e}_{\mu\nu}\,\mathrm{d}z + \int_{d}^{t/2} \bar{L}_{\alpha\beta\mu\nu}\dot{e}_{\mu\nu}\,\mathrm{d}z \\ \overset{\text{(I)}}{\dot{M}}_{\alpha\beta} &= \int_{-t/2}^{d} \bar{\mathscr{L}}_{\alpha\beta\mu\nu}\dot{e}_{\mu\nu}z\,\mathrm{d}z + \int_{d}^{t/2} \bar{L}_{\alpha\beta\mu\nu}\dot{e}_{\mu\nu}z\,\mathrm{d}z\end{aligned}\right\} \quad (\text{for } r \leqslant \zeta R), \tag{15}$$

where "I" represents the inner field.

As ζR extends with the increase of the post-bifurcation loading, the sizes of the inner and outer field change continuously. However, for a specific loading moment, ζR is a fixed quantity. By substituting eqn (12) and eqn (15) into the equation of the virtual work principle, eqn (7), and performing integral transformations for $r \geqslant \zeta R$ and $r \leqslant \zeta R$, separately, we have the following three sets of equations.

(i) The governing equations of the inner field

$$\left\{\begin{aligned}&\tfrac{1}{12}[\overset{\text{(I)}}{\dot{\bar{M}}}_{r,\bar{r}\bar{r}} + (2\overset{\text{(I)}}{\dot{\bar{M}}}_{r,\bar{r}} - \overset{\text{(I)}}{\dot{\bar{M}}}_{\theta,\bar{r}})/\bar{r}] + [\overset{\text{(I)}}{\dot{\bar{N}}}_{r}\bar{w}_{,\bar{r}\bar{r}} + \overset{\text{(I)}}{\dot{\bar{N}}}_{\theta}\bar{w}_{,\bar{r}}/\bar{r} \\ &\qquad + \overset{\text{(I)}}{\bar{N}}_{r}\dot{\bar{w}}_{,\bar{r}\bar{r}} + \overset{\text{(I)}}{\bar{N}}_{\theta}\dot{\bar{w}}_{,\bar{r}}/\bar{r}] = 0, \\ &\overset{\text{(I)}}{\dot{\bar{N}}}_{r,\bar{r}} + (\overset{\text{(I)}}{\dot{\bar{N}}}_{r} - \overset{\text{(I)}}{\dot{\bar{N}}}_{\theta})/\bar{r} = 0, \\ &\overset{\text{(I)}}{\dot{\bar{M}}}_{r}(0) \neq \infty, \quad \overset{\text{(I)}}{\dot{\bar{M}}}_{\theta}(0) \neq \infty, \quad \dot{\bar{u}}(0) = 0, \quad \dot{\bar{w}}_{,\bar{r}\bar{r}\bar{r}}(0) = \dot{\bar{w}}_{,\bar{r}}(0) = 0\end{aligned}\right.$$

$$[0 \leqslant \bar{r} \leqslant \zeta]. \tag{16}$$

(ii) The governing equations of the outer field

$$\left\{\begin{aligned}&\tfrac{1}{12}[\overset{\text{(II)}}{\dot{\bar{M}}}_{r,\bar{r}\bar{r}} + (2\overset{\text{(II)}}{\dot{\bar{M}}}_{r,\bar{r}} - \overset{\text{(II)}}{\dot{\bar{M}}}_{\theta,\bar{r}})/\bar{r}] + [\overset{\text{(II)}}{\dot{\bar{N}}}_{r}\bar{w}_{,\bar{r}\bar{r}} + \overset{\text{(II)}}{\dot{\bar{N}}}_{\theta}\bar{w}_{,\bar{r}}/\bar{r} \\ &\qquad + \overset{\text{(II)}}{\bar{N}}_{r}\dot{\bar{w}}_{,\bar{r}\bar{r}} + \overset{\text{(II)}}{\bar{N}}_{\theta}\dot{\bar{w}}_{,\bar{r}}/\bar{r}] = 0, \\ &\overset{\text{(II)}}{\dot{\bar{N}}}_{r,\bar{r}} + (\overset{\text{(II)}}{\dot{\bar{N}}}_{r} - \overset{\text{(II)}}{\dot{\bar{N}}}_{\theta})/\bar{r} = 0, \\ &\dot{\bar{w}}_{,\bar{r}}(1) = \dot{\bar{w}}(1) = 0, \quad \bar{w}_{,\bar{r}}(1) = \bar{w}(1) = 0\end{aligned}\right.$$

$$[\zeta \leqslant \bar{r} \leqslant 1]. \tag{17}$$

(iii) The continuity conditions on $\bar{r} = \zeta$

$$\overset{(II)}{\dot{\bar{M}}}_r(\zeta^+) = \overset{(I)}{\dot{\bar{M}}}_r(\zeta^-), \quad \overset{(II)}{\dot{\bar{M}}}_{r,\bar{r}}(\zeta^+)\zeta - \overset{(II)}{\dot{\bar{M}}}_\theta(\zeta^+) = \overset{(I)}{\dot{\bar{M}}}_{r,\bar{r}}(\zeta^-)\zeta - \overset{(I)}{\dot{\bar{M}}}_\theta(\zeta^-),$$

$$\dot{\bar{w}}(\zeta^+) = \dot{\bar{w}}(\zeta^-), \quad \dot{\bar{w}}_{,\bar{r}}(\zeta^+) = \dot{\bar{w}}_{,\bar{r}}(\zeta^-), \quad \bar{w}(\zeta^+) = \bar{w}(\zeta^-),$$

$$\dot{\bar{u}}(\zeta^+) = \dot{\bar{u}}(\zeta^-), \quad \overset{(II)}{\dot{\bar{N}}}_r(\zeta^+) = \overset{(I)}{\dot{\bar{N}}}_r(\zeta^-), \quad \bar{u}(\zeta^+) = \bar{u}(\zeta^-). \tag{18}$$

In the formulae above

$$\bar{u} = uR/t^2, \quad \bar{w} = w/t, \quad \bar{r} = r/R, \quad \dot{\bar{e}}_{\alpha\beta} = \frac{R^2}{t^2}\dot{e}_{\alpha\beta},$$

$$\dot{\bar{N}} = \dot{N}_{\alpha\beta}\Big/\left[\frac{\bar{E}_c}{1-\bar{v}_c^2}\frac{t^3}{R^2}\right], \quad \dot{\bar{M}}_{\alpha\beta} = \dot{M}_{\alpha\beta}\Big/\left[\frac{t^4}{12R^2}\frac{\bar{E}_c}{1-\bar{v}_c^2}\right] \tag{19}$$

are nondimensional quantities and ζ^- and ζ^+ in eqn (18) correspond to the approaching of ζ from the inner and outer fields, respectively.

The boundary between the inner field and the outer field, $\bar{r} = \zeta$, is defined by setting $d = -t/2$ in eqn (14),

$$[\dot{\bar{u}}_{,\bar{r}} + \bar{w}_{,\bar{r}}\dot{\bar{w}}_{,\bar{r}} + \dot{\bar{u}}/\bar{r} + \tfrac{1}{2}(\dot{\bar{w}}_{,\bar{r}\bar{r}} + \dot{\bar{w}}_{,\bar{r}}/\bar{r})]|_{\bar{r}=\zeta} = 0. \tag{20}$$

Obviously, the governing equations of the post-bifurcation problem, eqn (14) and eqns (16)–(20), constitute a moving boundary problem.

3. PLASTIC POST-BIFURCATION ANALYSIS

Elastic unloading plays a crucial role in plastic post-bifurcation analysis. Its existence and extension change the governing equations of a structure from a fixed boundary problem to a moving boundary problem. When we concentrate on the initial stage of post-bifurcation deformation, the value of ζ, which characterizes the size of the unloading area, is small. The inner field behaves like a boundary layer. To obtain a solution in the boundary layer, an expansion of scale is required. On the other hand, the continuity condition requires a systematic transformation in the outer field as well. Following the scheme of Su and Lu (1990b), we complete the plastic post-bifurcation analysis in three steps.

In the following asymptotic analysis, ζ is used as the perturbation parameter and the time. The discussions are based on the nondimensional quantities as defined in eqn (19). The bars over the nondimensional quantities are omitted for simplicity.

3.1. *Boundary layer analysis of the inner field*

The governing equations are shown in eqn (14) and eqn (16).

The boundary layer transformation

$$\xi = r/\zeta, \quad \Rightarrow r = 0, \quad \xi = 0; \quad r = \zeta, \quad \xi = 1 \tag{21}$$

changes the extending area $r \in [0, \zeta]$ to the fixed region, $\xi \in [0, 1]$. The governing equations in terms of the variable r, can be transformed into equations in terms of variable ξ, accordingly.

Since the unloading area of the bifurcated circular plate is developed from a point, the lowest non-zero order terms in the expansions of velocities $\dot{u}$, $\dot{w}$ should be of the order five (Su, 1988); so we expand

$$\dot{w}(r,\zeta) = \overset{(I)}{\dot{w}}(\xi,\zeta) = \overset{(I)}{\dot{w}}_6(\xi)\zeta^5 + \overset{(I)}{\dot{w}}_7(\xi)\zeta^6 + \cdots \tag{22}$$

$$\dot{u}(r,\zeta) = \overset{(I)}{\dot{u}}(\xi,\zeta) = \overset{(I)}{\dot{u}}_6(\xi)\zeta^5 + \overset{(I)}{\dot{u}}_7(\xi)\zeta^6 + \cdots \tag{23}$$

$$\bar{L}_{\alpha\beta\mu\nu} = \bar{L}^{c}_{\alpha\beta\mu\nu} + \left.\frac{\partial \bar{L}_{\alpha\beta\mu\nu}}{\partial \sigma_{\delta\gamma}}\right|_{\sigma=\sigma_c} (\sigma_{\delta\gamma} - \sigma^{c}_{\delta\gamma}) + \cdots \tag{24}$$

$$\delta(\xi, \zeta) = d_1(\xi)\zeta + d_2(\xi)\zeta^2 + \cdots, \tag{25}$$

where δ is the length of the unloading interval in the thickness

$$\delta = d/t + 1/2. \tag{26}$$

Note that as $\delta = 0$ for $r = \zeta$, d_n in eqn (25) should satisfy

$$d_n(1) = 0, \quad (n \geqslant 1). \tag{27}$$

Substituting the above expansions into the transformed governing equations which are expressed in terms of the variable ξ, and using the formulae for $\overset{(I)}{\dot{N}}_{\alpha\beta}$ and $\overset{(I)}{\dot{M}}_{\alpha\beta}$ in eqn (15), we can obtain perturbation equations by following the conventional asymptotic procedure. The solutions to these perturbation equations are

$$\begin{aligned} &\overset{(I)}{w}_6(\xi) = A_6 + B_6\xi^2, && \overset{(I)}{w}_7(\xi) = A_7 + B_7\xi^2, \ldots \\ &\overset{(I)}{u}_6(\xi) = 0, && \overset{(I)}{u}_7(\xi) = -B_8\xi, \ldots \\ &d_1(\xi) = 0, && d_2(\xi) = \frac{k^2}{8}(1 - \xi^2), \ldots. \end{aligned} \tag{28}$$

Since the boundary conditions on $\xi = 1$ are not given, there are two unknowns in the solutions of each order asymptotic analysis.

3.2. *The asymptotic analysis of the outer field*

The governing equations of the outer field are shown in eqn (17).

The systematic transformation

$$\bar{\xi} = \frac{r - \zeta}{1 - \zeta}, \quad \Rightarrow r = \zeta, \quad \bar{\xi} = 0; \quad r = 1, \quad \bar{\xi} = 1 \tag{29}$$

transforms the shrinking region $r \in [\zeta, 1]$, to the fixed region $\bar{\xi} \in [0, 1]$. Expand rates of displacements as

$$\dot{w}(r, \zeta) = \overset{(II)}{\dot{w}}(\bar{\xi}, \zeta) = \overset{(II)}{w}_6(\bar{\xi})\zeta^5 + \overset{(II)}{w}_7(\bar{\xi})\zeta^6 + \cdots \tag{30}$$

$$\dot{u}(r, \zeta) = \overset{(II)}{\dot{u}}(\bar{\xi}, \zeta) = \overset{(II)}{u}_6(\bar{\xi})\zeta^5 + \overset{(II)}{u}_7(\bar{\xi})\zeta^6 + \cdots. \tag{31}$$

Here the method of Solution–Transformation (Su and Lu, 1990b) is used for the solutions of $\overset{(II)}{\dot{w}}$ and $\overset{(II)}{\dot{u}}$. The field functions in terms of the original variable r are expanded first as

$$\dot{w}(r, \zeta) = w_6(r)\zeta^5 + w_7(r)\zeta^6 + \cdots \tag{32}$$

$$\dot{u}(r, \zeta) = u_6(r)\zeta^5 + u_7(r)\zeta^6 + \cdots. \tag{33}$$

By solving the governing equations in eqn (17) with $\dot{u}(r)$ and $\dot{w}(r)$ expanded as in eqn (32)–eqn (33) using perturbation, we obtain (Su, 1988)

$$\dot{w}(r,\zeta) = [\dot{a}_{\rm I}(\zeta)+\dot{a}_{\rm II}(\zeta)][J_0(kr)-J_0(k)] + \frac{\beta(\zeta)}{2k^3}\left\{rJ_1(kr)+\frac{J_0(k)}{Y_1(k)}[Y_0(kr)-Y_0(k)]\right\}+O(\zeta^{16}) \quad (34)$$

$$\dot{u}(r,\zeta) = \dot{b}_{\rm I}(\zeta)r+O(\zeta^{10}), \quad (35)$$

where Y_n is the nth-order Neumann function, and

$$\dot{a}_{\rm I}(\zeta) = a_6\zeta^5+a_7\zeta^6+\cdots+a_{11}\zeta^{10} \quad (36)$$

$$\dot{a}_{\rm II}(\zeta) = a_{12}\zeta^{11}+a_{13}\zeta^{12}+\cdots+a_{17}\zeta^{16} \quad (37)$$

$$\dot{b}_{\rm I}(\zeta) = b_6\zeta^5+b_7\zeta^6+\cdots+b_{11}\zeta^{10} \quad (38)$$

$$\beta(\zeta) = \left[12(1+\bar{v}_c)k^2+(2\psi_1+\psi_2)(1+\bar{v}_c)\frac{t^2}{R^2}k^4\right](a_{\rm I}b_{\rm I})^{\cdot}+\frac{t^2}{R^2}\frac{\psi_2}{2}(1-\bar{v}_c)k^4\dot{b}_{\rm I}a_{\rm I} \quad (39)$$

where $a_{\rm I}$, $a_{\rm II}$ and $b_{\rm I}$ are the integrations of $\dot{a}_{\rm I}$, $\dot{a}_{\rm II}$ and $\dot{b}_{\rm I}$, and

$$\psi_1 = -\psi_2\left[\frac{1}{2}+\frac{1}{12}\left(\frac{1+v}{1-v}\right)\frac{\sigma_c}{T-T_t^c}\left.\frac{\mathrm{d}T_t}{\mathrm{d}\sigma}\right|_{\sigma=\sigma_c}\right], \quad \psi_2 = -3(T-T_t^c)\frac{1-v}{(1+v)\sigma_c}$$

$$T = E/(1-v), \quad T_t = \bar{E}/(1-\bar{v}). \quad (40)$$

Changing the variable r in the solutions of $\dot{u}$ and $\dot{w}$ in eqns (34)–(35) to the variable $\bar{\xi}$, by using eqn (29) and regrouping $\dot{u}$ and $\dot{w}$ in the form of eqns (30)–(31), provides

$$\overset{\rm (II)}{u}_6(\bar{\xi}) = b_6\bar{\xi},\ldots \quad (41)$$

$$\overset{\rm (II)}{w}_6(\bar{\xi}) = a_6[J_0(k\bar{\xi})-J_0(k)],$$

$$\overset{\rm (II)}{w}_7(\bar{\xi}) = a_7[J_0(k\bar{\xi})-J_0(k)]+a_6k(\bar{\xi}-1)J_1(k\bar{\xi})$$

$$\ldots, \quad (42)$$

where a_n and b_n are determined by continuity conditions on $\bar{\xi}=0$.

Since $\dot{w}(r)$ is singular at $r=0$, $\overset{\rm (II)}{w}_n(\bar{\xi})$ $(n \geqslant 12)$, from the procedure above will include the terms of $\ln\zeta$. This is incorrect, because $\overset{\rm (II)}{w}_n(\bar{\xi})$ is defined as a function of $\bar{\xi}$ only by the expansion in eqn (30). Strictly speaking, $\overset{\rm (II)}{w}$ should be expanded in a series of ζ^n, $\zeta^n\ln\zeta$ $(n = 1,\ldots)$. However, for our problem $\overset{\rm (II)}{w}_n$, $n \geqslant 12$, is used only for the purpose of determining $\overset{\rm (II)}{w}_n$, $n < 12$. The terms of $\ln\zeta$ will be eliminated and therefore, the final results will not be affected.

3.3. *The connection of the inner and outer field*

In terms of the variables ξ and $\bar{\xi}$, the continuity conditions in eqn (18) are conditions for the inner and outer field solutions when $\xi = 1$ and $\bar{\xi} = 0$, respectively. The unknowns in the two field solutions a_n, b_n and A_n, B_n can be determined by substituting solutions of the respective field into eqn (18), and comparing the terms of equal powers of ζ in each equation.

The final results obtained by above procedure are

$$
w = \begin{cases}
\frac{a_6}{6}[1-J_0(k)]\zeta^6 + \left\{\frac{a_8}{8}[1-J_0(k)] - \frac{k^2}{24}a_6\xi^2\right\}\zeta^8 + \left\{\frac{a_{10}}{10}[1-J_0(k)] - \frac{k^2}{32}a_8\xi^2 \right. \\
\left. \quad + \frac{3}{48}\frac{k^4}{4!}a_6\xi^4\right\}\zeta^{10} + \left\{\frac{a_{12}}{12}[1-J_0(k)] + a_{12}^* + \left[\frac{3k^6 a_6}{1280}(m-1) - \frac{a_{10}}{20}k^2\right]\xi^2 \right. \\
\left. \quad + \frac{3}{64}\cdot\frac{1}{4!}\left[-\frac{3k^6}{8}a_6(m-1) + a_8 k^4\right]\xi^4 + \frac{a_6 k^6}{24^3}[3(m-1)-1]\xi^6\right\}\zeta^{12} + \cdots \\
\qquad (0 \leqslant r \leqslant \zeta \Leftrightarrow 0 \leqslant \xi \leqslant 1) \quad (43) \\
[1-J_0(k)]\left\{\frac{a_6}{6}\zeta^6 + \frac{a_8}{8}\zeta^8 + \cdots\right\}\frac{J_0(kr)-J_0(k)}{1-J_0(k)} + \frac{\{\beta_{12}\zeta^{12}+\cdots\}}{2k^3} \\
\quad \times \left\{J_1(kr)r + \frac{J_0(k)}{Y_1(k)}[Y_0(kr) - Y_0(k)]\right\} + \cdots \quad (\zeta \leqslant r \leqslant 1 \Leftrightarrow 0 \leqslant \bar{\xi} \leqslant 1) \quad (44)
\end{cases}
$$

$$
u = u_c(r) + \left[\frac{a_6}{24}k^2\zeta^6 + \frac{k^2}{32}\left(a_8 - \frac{a_6}{4}k^2\right)\zeta^8 + \frac{k^2}{40}\left(a_{10} - \frac{a_8}{4}k^2 + \frac{a_6}{64}k^4\right)\zeta^{10} + \cdots\right]r + \cdots \tag{45}
$$

$$
\sigma/\sigma_c = 1 - 3(1+\bar{v}_c)\left[\frac{a_6}{6}\zeta^6 + \frac{1}{8}\left(a_8 - \frac{a_6}{4}k^2\right)\zeta^8 + \frac{1}{10}\left(a_{10} - \frac{a_8}{4}k^2 + \frac{a_6}{64}k^4\right)\zeta^{10} + \cdots\right] + \cdots \tag{46}
$$

$$
a_6 = \frac{\pi k^5}{16^2}\frac{Y_1(k)}{J_0(k)}\frac{T-T_t^c}{T_t^c} \Bigg/ \left\{1 - \frac{k^2}{24}\frac{t^2}{R^2}\frac{\mathrm{d}T_t}{\mathrm{d}\sigma}\bigg|_{\sigma=\sigma_c} + \frac{3}{4}\frac{1-\bar{v}_c}{1+v}(1-v)\frac{T-T_t^c}{T_t^c}\right\} \tag{47}
$$

$$
a_8 = \tfrac{1}{3}a_6 k^2 - \tfrac{1}{2}a_6 k^2(\bar{m}_c - 1) + \frac{8}{\pi k^3}\frac{(1-\bar{v}_c)(1-v)}{1+v}\frac{J_0(k)}{Y_1(k)}a_6^2, \tag{48}
$$

where a_{12}^* is a constant, and

$$
m = \frac{E}{1-v^2}\frac{1-\bar{v}^2}{\bar{E}}, \quad \Rightarrow \bar{m}_c = \frac{E}{1-v^2}\cdot\frac{1-\bar{v}_c^2}{\bar{E}_c} \tag{49}
$$

$$
\beta_{12} = \left[24(1+\bar{v}_c)k^2 + 2(2\psi_1+\psi_2)(1+\bar{v}_c)\frac{t^2}{R^2}k^4 + \frac{t^2}{R^2}\frac{\psi_2}{2}(1-\bar{v}_c)k^4\right]\frac{a_6^2}{24}\frac{k^2}{12}. \tag{50}
$$

4. IMPERFECTION SENSITIVITY ANALYSIS

All the quantities in this section are nondimensional as shown in eqn (19). The imperfection $\bar{w}$ is nondimensionalized by t. The bars on the nondimensional quantities are omitted.

The imperfections discussed are in the form of

$$
\bar{w} = -\varepsilon[J_0(kr) - J_0(k)]/[1-J_0(k)]. \tag{51}
$$

If ε is infinitesimal, the plate will deflect only slightly in a fairly large range of loading. This range can include the period when the whole plate deforms plastically, until unloading starts at the point $(0, 0, -t/2)$. The deformation of the plate in the range will be approximated by a homogeneous one as if the plate were perfect.

The deformation of the plate before the occurrence of unloading is a hypoelastic one. Hutchinson (1974) presented formulas for the load $\hat{\sigma}$, and the deflection $\hat{w}$, when the unloading just begins:

$$\hat{w} = -\hat{\xi}[J_0(kr) - J_0(k)]/[1 - J_0(k)] \tag{52}$$

$$\hat{\xi} = [\lambda_c \rho/\lambda_1]^{1/2} \varepsilon^{1/2} \tag{53}$$

$$\hat{\sigma}/\sigma_c = 1 - [\lambda_1 \rho/\lambda_c]^{1/2} \varepsilon^{1/2}, \tag{54}$$

where λ_c, λ_1 and ρ are parameters defined by plastic bifurcation analysis (Hutchinson, 1974). It is noted that $\hat{\xi}$ and $\hat{\sigma} - \sigma_c$ are proportional to $\varepsilon^{1/2}$.

Since the plate is assumed to deform approximately like a perfect one before unloading up to $\hat{\sigma}$, we can write

$$\dot{\sigma}_r = \frac{\hat{\bar{E}}}{1-\hat{\bar{\nu}}^2}(\dot{e}_r + \hat{\bar{\nu}}\dot{e}_\theta), \quad \dot{\sigma}_\theta = \frac{\hat{\bar{E}}}{1-\hat{\bar{\nu}}^2}(\dot{e}_\theta + \hat{\bar{\nu}}\dot{e}_r) \tag{55}$$

where $\hat{\bar{E}}$, $\hat{\bar{\nu}}$ are effective moduli for $\sigma_r = \sigma_\theta = \hat{\sigma}$.

When $\sigma > \hat{\sigma}$, the unloading area begins to extend from the point $(0, 0, -t/2)$. The basic equations governing the post-unloading deformation can be obtained in the same way as shown in Section 2 for the post-bifurcation analysis. Nearly all the equations are the same. The only change is that the strain rate in the radial direction is affected by the imperfection

$$\dot{e}_r = \dot{u}_{,r} + w_{,r}\dot{w}_{,r} + \tilde{w}_{,r}\dot{w}_{,r} - z\dot{w}_{,rr}, \tag{56}$$

and the equilibrium eqns (16) and (17) should have $(\overset{(\mathrm{I})}{\dot{N}_r}\tilde{w}_{,rr} + \overset{(\mathrm{I})}{\dot{N}_\theta}\tilde{w}_{,r}/r)$ and $(\overset{(\mathrm{II})}{\dot{N}_r}\tilde{w}_{,rr} + \overset{(\mathrm{II})}{\dot{N}_\theta}\tilde{w}_{,r}/r)$ added to their left sides, respectively.

The governing equations for the post-unloading analysis of the imperfect plate also constitute a moving boundary problem. The asymptotic analysis of Section 3 can be carried out in almost the same way, with minor adaptations as explained in the following.

First, the post-unloading deformation is not only related to the loading, but also to the deflection of the first unloading $\hat{w}$, or $\varepsilon^{1/2}$ by eqn (53). So, a field function $f(r, \zeta, \varepsilon^{1/2})$ is of two parameters, and should be expanded into

$$f(r, \zeta, \varepsilon^{1/2}) = f_6^{\mathrm{i}}(r)\zeta^5 + f_7^{\mathrm{i}}(r)\zeta^6 + \cdots + \varepsilon^{1/2}[f_1^{\mathrm{ii}}(r) + f_2^{\mathrm{ii}}(r)\zeta + \cdots] + \varepsilon[f_1^{\mathrm{iii}}(r) + f_2^{\mathrm{iii}}(r)\zeta + \cdots] + \cdots, \tag{57}$$

where r should be replaced by ξ for the inner field, and $\bar{\xi}$ for the outer field when f is the function of the respective field. Also, ε is infinitesimally small by definition. When ε appears together with ζ in the asymptotic analysis after substituting expansions of eqn (57) into the governing equations, it is difficult to determine asymptotic orders. So we write ε as

$$\varepsilon = \alpha^2 \zeta^{12} \tag{58}$$

and substitute it into eqn (57) to get

$$f = f_6^{\mathrm{i}}(r)\zeta^5 + [f_7^{\mathrm{i}}(r) + f_1^{\mathrm{ii}}(r)\alpha]\zeta^6 + \cdots. \tag{59}$$

When expansions of the above form are applied to the governing equations, perturbation equations can be obtained with α being regarded as a fixed quantity. The perturbation equations are in the form of

$$\phi^n[f_n^{\mathrm{i}}, f_{n-1}^{\mathrm{i}}, \ldots, f_{n-6}^{\mathrm{ii}}, f_{n-7}^{\mathrm{ii}}, \ldots] + \alpha\psi^n[f_n^{\mathrm{i}}, f_{n-1}^{\mathrm{i}}, \ldots, f_{n-6}^{\mathrm{ii}}, f_{n-7}^{\mathrm{ii}}, \ldots] = 0 \tag{60}$$

where ϕ^n and ψ^n are linear operators. Since the imperfection may be of different sizes, the α can be treated as an independent variable proportional to $\varepsilon^{1/2}$. This leads to

$$\phi^n[f_n^{\mathrm{i}}, f_{n-1}^{\mathrm{i}}, \ldots, f_{n-6}^{\mathrm{ii}}, \ldots] = 0, \quad \psi^n[f_n^{\mathrm{i}}, \ldots, f_{n-6}^{\mathrm{ii}}, \ldots] = 0. \tag{61}$$

These are the perturbation equations when the imperfections are included.

Next, as a result of the assumption that the plate deforms homogeneously when $\sigma = \hat{\sigma}$, the plastic moduli denoted by $\hat{L}_{\alpha\beta\mu\nu}$ are equal everywhere in the plate. The plastic moduli after unloading can be expanded as

$$\bar{L}_{\alpha\beta\mu\nu} = \hat{\bar{L}}_{\alpha\beta\mu\nu} + \left.\frac{\partial \bar{L}_{\alpha\beta\mu\nu}}{\partial \sigma_{\delta\gamma}}\right|_{\sigma=\hat{\sigma}} (\sigma_{\delta\gamma} - \hat{\sigma}_{\delta\gamma}) + \cdots. \tag{62}$$

When the above expansion is applied in the asymptotic analysis, the lowest order perturbation equation of the outer field is reduced to

$$\hat{\sigma} = -\frac{1}{12}\frac{\hat{\bar{E}}}{1-\hat{\bar{\nu}}^2}\frac{t^2}{R^2}k^2 - \frac{k^2t^2}{4R^2}\frac{\hat{\bar{E}}}{1-\hat{\bar{\nu}}^2}\xi\frac{1}{1-J_0(k)}. \tag{63}$$

It can be seen from eqns (53)–(54) that $\hat{\sigma}$ and ξ do not satisfy eqn (63). It is assumed that eqns (53)–(54) and eqn (63) are approximately compatible once ε is small enough, so that higher order calculations can be carried out.

Finally, it is usually cumbersome to calculate $\hat{\bar{L}}_{\alpha\beta\mu\nu}$, $\partial\bar{L}_{\alpha\beta\mu\nu}/\partial\sigma_{\delta\gamma}|_{\sigma=\hat{\sigma}}$, etc. since they are related to ε. On the other hand, we notice $\hat{\bar{L}}_{\alpha\beta\mu\nu} - \bar{L}^c_{\alpha\beta\mu\nu} = O(\zeta^6)$. After getting the final results of the imperfection sensitivity analysis, we use $\bar{L}^c_{\alpha\beta\mu\nu}$, etc. in the place of $\hat{\bar{L}}_{\alpha\beta\mu\nu}$, etc. This simplifies the calculation.

The results of the imperfection sensitivity analysis of clamped circular plates are

$$w(0) = -\xi + [1 - J_0(k)]\left\{\frac{a_6^{\mathrm{i}}}{6}\zeta^6 + \frac{a_8^{\mathrm{i}}}{8}\zeta^8 + \frac{a_2^{\mathrm{ii}}}{2}\varepsilon^{1/2}\zeta^2 + \cdots\right\} \tag{64}$$

$$\sigma = \hat{\sigma} + \frac{1}{12}\frac{\hat{\bar{E}}}{1-\hat{\bar{\nu}}}\frac{t^2}{R^2}k^2\left\{\frac{a_6^{\mathrm{i}}}{2}\zeta^6 + \frac{3}{8}\left(a_8^{\mathrm{i}} - \frac{k^2}{4}a_6^{\mathrm{i}}\right)\zeta^8 + \frac{3}{2}a_2^{\mathrm{ii}}\varepsilon^{1/2}\zeta^2 + \cdots\right\}, \tag{65}$$

where $w(0)$ is the deflection of the plate center, and

$$a_6^{\mathrm{i}} = \frac{\pi k^5}{16^2}\frac{Y_1(k)}{J_0(k)}\frac{T-\hat{T}_t}{\hat{T}_t} \Bigg/ \left\{1 - \frac{k^2}{24}\frac{t^2}{R^2}\left.\frac{\mathrm{d}T_t}{\mathrm{d}\sigma}\right|_{\sigma=\hat{\sigma}} + \frac{3}{4}\frac{1-\hat{\bar{\nu}}}{1+\nu}(1-\nu)\frac{T-\hat{T}_t}{\hat{T}_c}\right\}$$

$$a_8^{\mathrm{i}} = \tfrac{1}{3}a_6^{\mathrm{i}}k^2 - \tfrac{1}{2}a_6^{\mathrm{i}}k^2(\hat{\bar{m}} - 1) + \frac{8}{\pi k^3}\frac{1-\nu}{1+\nu}(1-\hat{\bar{\nu}})\frac{J_0(k)}{Y_1(k)}(a_6^{\mathrm{i}})^2$$

$$a_2^{\mathrm{ii}} = \frac{32}{\pi k^3}\frac{J_0(k)}{Y_1(k)}\frac{a_6^{\mathrm{i}}}{\hat{\bar{m}} - 1}\left(-\frac{\xi}{\varepsilon^{1/2}}\right)\frac{1+\hat{\bar{\nu}}}{1-J_0(k)}$$

$$\hat{\bar{m}} = \frac{E}{1-\nu^2}\cdot\frac{1-\hat{\bar{\nu}}^2}{\hat{\bar{E}}}. \tag{66}$$

It is obvious that when $\hat{\bar{E}} = \bar{E}_c$, $\hat{\bar{m}} = \bar{m}_c$, we have

$$a_6^{\mathrm{i}} = a_6, \quad a_8^{\mathrm{i}} = a_8,$$

where a_6, a_8 are from eqns (47)–(48).

5. NUMERICAL EXAMPLE AND DISCUSSIONS

The plastic post-bifurcation behaviors of clamped circular plates are quite different from the elastic postbuckling behaviors of the same structures. This is clearly illustrated when the results of eqns (43)–(46) are expressed as expansions in terms of eigenmodal amplitudes. For this reason the eigenmodal amplitude is denoted by η, where

$$\eta = -[1-J_0(k)][\tfrac{1}{6}a_6\zeta^6+\tfrac{1}{8}a_8\zeta^8+\cdots] \tag{67}$$

from eqn (44). By substituting the inversion of eqn (67) into eqns (43)–(46), we get

$$w = \begin{cases} \left\{-\dfrac{J_0(kr)-J_0(k)}{1-J_0(k)}+\dfrac{a_6k^6}{4608}h_1^6(m-1)r^6+\cdots\right\}\eta+\cdots \quad (r \leqslant \zeta) & (68) \\ -\dfrac{J_0(kr)-J_0(k)}{1-J_0(k)}\eta+\left\{\dfrac{\beta_2h_1^{12}}{2k^3}\eta^2+\cdots\right\} & \\ \times\left\{J_1(kr)r+\dfrac{J_0(k)}{Y_1(k)}[Y_0(kr)-Y_0(k)]+\cdots\right\} \quad (r \geqslant \zeta) & (69) \end{cases}$$

$$\sigma/\sigma_c = 1+\lambda_1\eta+\lambda_2\eta^{4/3}+\lambda_3\eta^{5/3}+\cdots, \tag{70}$$

where

$$\lambda_1 = \frac{3(1+\bar{v}_c)}{1-J_0(k)}, \quad \lambda_2 = \frac{3(1+\bar{v}_c)}{32}a_6k^2h_1^8, \quad \lambda_3 = -3(1+\bar{v}_c)\left[\frac{a_8}{160}k^2+\frac{a_6k^4}{640}\right]h_1^{10}$$

$$h_1 = \{-6/[a_6-a_6J_0(k)]\}^{1/6}. \tag{71}$$

The result of eqn (69) shows that the second-order term of the post-bifurcation deflection expanded in terms of the eigenmodal amplitude is not zero, unlike that of elastic postbuckling (Thompson and Hunt, 1973). This is caused by the unloading in the center, as the singular Neumann function expresses itself.

A 5/3-order term immediately follows the 4/3-order term in the load–deflection relation. These high order terms will play important roles once the post-bifurcation deflection develops to a certain degree. This is in contrast to the case of the elastic postbuckling, where the load–deflection relation can be expanded in integral powers of the buckling mode amplitude. It is easy to understand that high order asymptotic terms are much more important in the plastic post-bifurcation than in the elastic postbuckling.

Another aspect of interest is that the first-order expansion of the deflection in the inner field is not the eigenmode itself, as in the case of elastic postbuckling. It seems to be a contradiction. Actually this is the natural outcome of a moving boundary problem. As a result of the extension of the unloading area, the governing equations are changing. The first-order asymptotic approximations to these equations are changing, too. In this regard, the validity of the assumption that the first-order terms in the expansions of the post-bifurcation deformation should be the eigenmode itself, is suspect as far as the plastic post-bifurcation is concerned (Su, 1990).

To further understand the plastic post-bifurcation and imperfection sensitivity behaviors of the clamped plate, a numerical example is given here.

The geometrical and material parameters of the example are the same as those adopted by Needleman (1975). The uniaxial strain–stress relation of the plate is

$$e/e_y = \begin{cases} \sigma/\sigma_y & (\sigma < \sigma_y), \\ \frac{1}{n}(\sigma/\sigma_y)^n + 1 - \frac{1}{n} & (\sigma > \sigma_y), \end{cases} \tag{72}$$

where e_y and σ_y are the yielding strain and stress, respectively, and $n = 12$. The ratio of plate thickness to radius is

$$(t/R)^2 = 24(1-v^2)e_y/k^2, \quad v = 1/3. \tag{73}$$

From eqns (72) and (73), we obtain

$$\sigma_c/\sigma_y = 1.123, \quad E/E_t^c = 3.5826, \quad \bar{E}_c = 0.6067E, \quad \bar{m}_c = 0.7847$$

$$T_t^c = 0.5107E, \quad \bar{v}_c = -0.1897, \quad e_y \left.\frac{\mathrm{d}T_t}{\mathrm{d}\sigma}\right|_{\sigma=\sigma_c} = -4.576.$$

The coefficients in eqns (64)–(65) are calculated after replacing $\hat{\bar{L}}_{\alpha\beta\mu\nu}$, etc. by $\bar{L}^c_{\alpha\beta\mu\nu}$, etc.

$$a_6 = a_6^{i} = -3.3892, \quad a_8 = a_8^{i} = 2.62, \quad a_2^{ii} = -0.144$$

$$\hat{\xi} = 0.3118\varepsilon^{1/2}, \quad \hat{\sigma} = \sigma_c - 0.5393\sigma_c\varepsilon^{1/2}.$$

So, the results of the imperfection sensitivity analysis are

$$w(0) = -0.3118\varepsilon^{1/2} - 0.101\varepsilon^{1/2}\zeta^2 - 0.7906\zeta^6 + 0.4593\zeta^8 + \cdots \tag{74}$$

$$\sigma/\sigma_c = 1 - 0.5393\varepsilon^{1/2} + 0.175\varepsilon^{1/2}\zeta^2 + 1.373\zeta^6 - 4.576\zeta^8 + \cdots. \tag{75}$$

In eqn (70), the results of the post-bifurcation analysis of the perfect plate become

$$\sigma/\sigma_c = 1 + 1.733\eta - 5.154\eta^{4/3} + 3.224\eta^{5/3}. \tag{76}$$

Its maximum is obtained when $\eta = 0.0412$, and is

$$\sigma_m = 1.014\sigma_c. \tag{77}$$

Hutchinson (1974) and Needleman (1975) calculated the load deflection relationship, truncated at 4/3-order for the circular plate, on the basis of Hutchinson's plastic post-bifurcation theory. Within the 4/3 order, their results are almost the same as ours. However, there are inherent difficulties in extending Hutchinson's general theory to higher order asymptotic analysis (Su, 1988, 1990). If the 5/3 term in eqn (76) is not included, the maximum is $\sigma_m = 1.007\sigma_c$, obtained at $\eta = 0.016$. Compared with Needleman's results

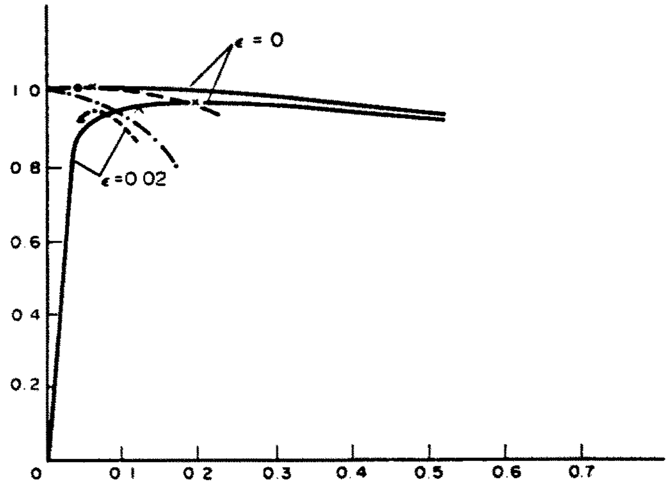

Fig. 2. Load-deflection relation. ———, result by finite element method (Needleman, 1975); ---- ---, result of present paper including 5/3 term; —·—·—·—, result not including 5/3 term; ▲, first unloading point from (52)–(54); ∧, first unloading point by finite element method; ●, maximum load of our analysis; ×, maximum load by finite element method.

obtained by finite element method, as shown in Fig. 2, the 5/3-order term improves the result. To be able to carry out the asymptotic post-bifurcation analysis to whatever high order, is one of the advantages of the method presented here.

If a_2^0 is ignored in eqns (74)–(75), the equations lead to

$$\begin{cases} w(0) = -0.3118\varepsilon^{1/2} - 0.7906\zeta^6 + 0.4593\zeta^8, \\ \sigma/\sigma_c = 1 - 0.5393\varepsilon^{1/2} + 1.373\zeta^6 - 4.576\zeta^8. \end{cases} \tag{78}$$

This is a conservative simplification. Comparing this with the post-bifurcation result, i.e. putting $\varepsilon = 0$ in eqns (74)–(75), we see that if σ_m is the maximum load of the perfect plate, the maximum for the imperfect plate $\bar{\sigma}_m$ from (78) will be

$$\bar{\sigma}_m = \hat{\sigma} + (\sigma_m - \sigma_c). \tag{79}$$

This can be of some significance in engineering. Once the post-bifurcation results, σ_m, and the first unloading load, $\hat{\sigma}$, are obtained, we can get an approximate maximum load of the imperfect plate from eqn (79) without going through the imperfection sensitivity analysis. The same conclusions in simple models and rectangular beams were presented in Su (1988).

Figure 2 shows the comparison between the author's results and Needleman's finite element results. Both results show that the geometrical imperfection lowers the load-bearing capacity of the plate significantly, even when the imperfection is infinitesimally small.

It can be found from Fig. 2 that there is some discrepancy between the first unloading point by eqns (52)–(54), and that by the finite element method. The discrepancy exists in the calculation of Hutchinson's model (Hutchinson, 1974; van der Heijden, 1979) too, though it is not as big as it is here. For this reason, van der Heijden (1979) concluded that the results from eqns (52)–(54) need refining. A good method that provides a more accurate first unloading point is, therefore, still a problem.

In the imperfection sensitivity analysis of Section 4, we assumed that the plate deforms homogeneously before unloading. This assumption may lead to large errors when the imperfection is not small enough. In this case, parts of the plate may even not deform in the plastic range while the first unloading begins at the center. However, the trends of the effects of the imperfections on the load-carrying capacity of the clamped plate are well reflected in our result. If the imperfection sensitivity curves by the present analysis and by Needleman's finite element analysis in Fig. 2 were translated to make the first unloading point coincide, the differences between the two calculations would be much smaller. This gives us some confidence. After all, as far as the authors know, the imperfection sensitivity analysis carried out here is the first attempt to bring the plastic post-bifurcation and imperfection sensitivity analysis into one analytical frame for real structures.

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